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by

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On The Efficient Calculation of Infra-red Fluxes and Cooling Rates Using the Two-Stream Equations

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Abstract

The calculation of infra-red radiative fluxes and cooling rates using the two-stream equations is discussed. It is argued that at infra-red wavelengths the two-stream equations are best viewed as an approximation to the differential radiance, the difference between the actual radiance and the local isotropic Planckian radiance. In this formulation the diffusivity factor and the forward and backward scattering fractions are related to the differential radiance. By considering the corresponding differential fluxes, approximate methods for representing overlapping gaseous absorption are derived. These methods offer substantial savings in computing time compared with an explicit representation of the overlap.

1 Introduction

In many meteorological applications of radiative transfer, particularly in general circulation models (GCMs), radiative fluxes and heating rates must be calculated rapidly but with reasonable accuracy. At solar wavelengths two-stream approximations are widely used for such purposes, but at infra-red wavelengths emissivity schemes are more commonly employed. These latter schemes are formulated using transmissions averaged across spectral bands and their computational cost is quadratic in the number of atmospheric layers. Conversely, the two-stream equations apply to monochromatic radiation and their computational cost is linear in the number of atmospheric layers; but to calculate the flux in a wide spectral interval a number of quasi-monochromatic calculations must be performed. Appropriate data for these calculations may be determined by *Exponential Sum fitting of Transmissivities* (ESFT) (Wiscombe and Evans, 1977) or from correlated k-distributions (Goody and Yung, 1989). Although it is expensive to represent overlapping gaseous absorption in this way, the linear dependence of the computational cost on the number of layers makes it desirable to re-examine the use of the two-stream approximation in the infra-red and to see whether more efficient means of treating the problem of overlapping gaseous absorption can be derived. Moreover, unlike emissivity schemes, two-stream approximations naturally include scattering. It will be suggested below that differential radiances and fluxes, that is, differences between the actual radiances or fluxes and the Planckian radiance or flux at the local temperature, are the most natural quantities for discussing two-stream approximations in the infra-red, enabling the physical basis of the approximations to be clarified and approximate but rapid methods for treating overlapping gaseous absorption to be derived.

Although Zdunkowski *et al* (1974) have given an appropriate general formulation of the two-stream equations for the infra-red region, which is further discussed by Zdunkowski *et al* (1980) and Zdunkowski and Korb (1985), there is still a common belief that only a limited class of two-stream approximations is phys-

ically consistent in this part of the spectrum (Toon *et al*, 1989). In fact, whilst leading to the correct formulation of the equations — a formulation that is consistent with standard emissivity schemes if a diffusivity factor of 1.66 is employed — Zdunkowski's derivation is consistent in its treatment of the Planckian source function only if a diffusivity factor of 2 is assumed. It will be shown in section 2 that the two-stream equations can be derived more satisfactorily by approximating the differential radiance rather than the full radiance. The diffusivity factor can now be seen to apply to the differential flux rather than the full flux, and therefore need not automatically be taken to be 2.

Recently, Ritter and Geleyn (1992) (hereafter RG92) proposed a method for reducing the computing time required to treat overlapping gaseous absorption with ESFT, which they called "Fast ESFT" (FESFT), by applying to the infra-red net fluxes a technique suggested in its original form by Geleyn and Hollingsworth (1979) for the calculation of short-wave fluxes. RG92 did not give a theoretical justification for the technique, but, as will be shown in section 3 below, its success can be explained by considering the differential fluxes. A slightly different but numerically more efficient form of this technique is derived. Whilst FESFT is usually a good approximation, it does exhibit some potentially undesirable features in the upper atmosphere. An alternative method of treating gaseous overlaps which involves the use of FESFT to construct an equivalent extinction is then discussed. In the final discussion the characteristics of these methods are compared and their possible application with correlated k-distributions is considered briefly.

The purpose of this paper is to examine the theoretical basis of these techniques and the calculations presented here are principally illustrative. Applications to standard test problems and comparisons with line-by-line models are presented in Edwards and Slingo (1995), where a description of the radiation code used may be found. This latter paper also contains a study of the effects of scattering in the infra-red where it is shown that, because of the optical properties of the scatterers and the frequency distribution of the radiation, the strongest signature of scattering is to be found in the partial window between the 15-μm CO_2 band and the rotation band, thus providing further motivation for using a two-stream approximation in the infra-red region. In the calculations presented here the atmospheric profiles of temperature and the mixing ratios of water vapour and ozone have been taken from the mid-latitude summer atmosphere used in the Inter-Comparison of Radiation Codes in Climate Models (Ellingson *et al*, 1991) and the mixing ratios of well-mixed gases have been taken as those applying in 1990 (Houghton *et al*, 1990). The atmosphere has been divided into 122 layers as in Schwartzkopf and Fels (1991) in order to illustrate the effects of the approximations clearly: in a GCM one would normally expect to use rather fewer layers.

2 The Two-stream Equations in the Infra-red

In this section, the two-stream equations in the infra-red region are derived by considering the differential radiances. The notation introduced here will be used in the following sections.

The equation of transfer for the monochromatic infra-red radiance, $I(\mathbf{n})$, in a direction, $\mathbf{n}$, at a height, z, in a plane-parallel atmosphere is

$$\frac{\mu}{\rho}\frac{dI}{dz} = -(\kappa_a + \kappa_s)I + \kappa_a B + \frac{\kappa_s}{4\pi}\int_\Omega P(\mathbf{n}', \mathbf{n})I(\mathbf{n}')\, d\omega_{\mathbf{n}'}, \tag{1}$$

where μ is the cosine of the angle between the unit vector $\mathbf{n}$ and the upward vertical, ρ is the density, B is the Planckian radiance, P is the scattering phase function, and κ_a and κ_s are the mass extinction coefficients for absorption and scattering respectively. Ω denotes all directions and Ω_+ and Ω_- will be used to represent the upward and downward hemispheres respectively. The mean free path of a photon is $1/(\rho(\kappa_a + \kappa_s))$ and in the optically thick limit, where the mean free path is much smaller than the vertical length-scale in the medium, $I \to B$. Consequently, the upward flux, F^+, and the downward flux, F^-, defined by the equation

$$F^\pm = \pm\int_{\Omega_\pm} \mu I\, d\omega_{\mathbf{n}} = 2\pi\int_0^{\pm 1} \mu I\, d\mu, \tag{2}$$

approach πB in this limit and the diffusivity factors for upward and downward radiation,

$$D_\pm = \pm\int_{\Omega_\pm} I\, d\omega_{\mathbf{n}} \Big/ \int_{\Omega_\pm} \mu I\, d\omega_{\mathbf{n}}, \tag{3}$$

both tend towards 2.

The two-stream equations are derived by integrating (1) over Ω_+ and Ω_-. To obtain equations involving fluxes alone, an assumption about angular distribution of the radiance must be made: this determines $D_\pm$. For the almost isotropic radiance in an opaque medium, $D_\pm$ must be 2: as noted by Toon *et al* (1989), any other choice for the diffusivity factors is physically inconsistent and will yield incorrect results.

However, where the radiance is almost isotropic, it is more useful to introduce the differential radiance

$$I' = I - B, \tag{4}$$

so that I' is a small correction to the Planckian radiance. Using I' the differential fluxes,

$$G^\pm = \pm\int_{\Omega_\pm} \mu I'\, d\omega_{\mathbf{n}}, \tag{5}$$

may be defined: these will be central to the discussion in the next section. To formulate the two-stream equations, (1) is integrated over $\Omega_\pm$ in the usual way,

but now terms involving B and I' are treated separately:

$$\begin{aligned}\frac{1}{\rho}\frac{d}{dz}\left[\int_{\Omega_\pm}\mu B\,d\omega_{\mathbf{n}}+\int_{\Omega_\pm}\mu I'\,d\omega_{\mathbf{n}}\right] &= -(\kappa_a+\kappa_s)\left[\int_{\Omega_\pm}B\,d\omega_{\mathbf{n}}+\int_{\Omega_\pm}I'\,d\omega_{\mathbf{n}}\right] \\ &+ \kappa_a\int_{\Omega_\pm}B\,d\omega_{\mathbf{n}} \\ &+ \frac{\kappa_s}{4\pi}\left[\int_{\Omega_\pm}\int_{\Omega}B\,P(\mathbf{n}',\mathbf{n})\,d\omega_{\mathbf{n}'}\,d\omega_{\mathbf{n}}\right. \\ &+ \left.\int_{\Omega_\pm}\int_{\Omega}I'(\mathbf{n}')P(\mathbf{n}',\mathbf{n})\,d\omega_{\mathbf{n}'}\,d\omega_{\mathbf{n}}\right].\end{aligned} \tag{6}$$

The terms involving B may be evaluated directly since B is isotropic: only terms involving I' need be approximated. To simplify these terms we redefine the diffusivity factors from the upward and downward differential radiances:

$$D_\pm = \pm\int_{\Omega_\pm}I'\,d\omega_{\mathbf{n}}\Big/\int_{\Omega_\pm}\mu I'\,d\omega_{\mathbf{n}}. \tag{7}$$

Again considering the differential radiances, forward and backward scattering fractions for upward radiation, α_+ and β_+, and corresponding fractions for downward radiation, α_- and β_-, may be defined:

$$\alpha_\pm = \int_{\Omega_\pm}\int_{\Omega_\pm}I'(\mathbf{n}')P(\mathbf{n}',\mathbf{n})\,d\omega_{\mathbf{n}'}\,d\omega_{\mathbf{n}}\Big/4\pi\int_{\Omega_\pm}I'(\mathbf{n}')\,d\omega_{\mathbf{n}'} \tag{8}$$

and

$$\beta\pm = \int_{\Omega_\mp}\int_{\Omega_\pm}I'(\mathbf{n}')P(\mathbf{n}',\mathbf{n})\,d\omega_{\mathbf{n}'}\,d\omega_{\mathbf{n}}\Big/4\pi\int_{\Omega_\pm}I'(\mathbf{n}')\,d\omega_{\mathbf{n}'}. \tag{9}$$

(Conservation of energy requires that $\alpha_\pm+\beta_\pm=1$). Equation (6) may therefore be written as

$$\pm\frac{1}{\rho}\frac{dG^\pm}{dz} = -D_\pm(\kappa_a+\kappa_s(1-\alpha_\pm))G^\pm + D_\mp\kappa_s\beta_\mp G^\mp \mp \frac{1}{\rho}\frac{d\pi B}{dz}. \tag{10}$$

By choosing the upper or the lower signs, the two-stream equations for the upward and downward differential fluxes respectively may be deduced from this equation. To apply these equations practically, the explicit forms must be given for $D_\pm$, $\alpha_\pm$ and $\beta_\pm$. If, as is normal, the phase function is expanded in Legendre functions and truncated after the first non-isotropic term, $\alpha_\pm$ and $\beta_\pm$ can be expressed in terms of $D_\pm$; for, making use of the azimuthal symmetry of I', and using g to denote the asymmetry factor,

$$\begin{aligned}\int_{\Omega_\pm}\int_{\Omega_\pm}I'(\mathbf{n}')P(\mathbf{n}',\mathbf{n})\,d\omega_{\mathbf{n}'}\,d\omega_{\mathbf{n}} &= 2\pi\int_{\Omega_\pm}I'(\mu')\,d\omega_{\mathbf{n}'} \\ &+ 3g\int_{\Omega_\pm}\mu\,d\omega_{\mathbf{n}}\int_{\Omega_\pm}\mu' I'(\mu')\,d\omega_{\mathbf{n}'} \\ &= 2\pi\int_{\Omega_\pm}I'(\mu')\,d\omega_{\mathbf{n}'}\left[1+\frac{3g}{2D_\pm}\right],\end{aligned} \tag{11}$$

where the first $\pm$ on the last line, subscripting Ω, refers to the domain of integration with respect to $\mathbf{n}'$ and the subscript of D to the domain of integration with respect to $\mathbf{n}$. Hence,

$$\alpha_{\pm} = 1/2 + 3g/(4D_{\pm}) \tag{12}$$

and similarly

$$\beta_{\pm} = 1/2 - 3g/(4D_{\pm}). \tag{13}$$

Provided that the phase function is so truncated, it follows that a two-stream approximation is effectively determined by the choice of $D_{\pm}$. We shall return to this point shortly, but it is convenient first to cast the equations in their final forms.

To obtain the standard form of the two-stream equations we take $D_+ \sim D_- = D$, and may therefore drop the suffixes on α and β. Then, defining the optical depth, τ, from the equation

$$d\tau = -(\kappa_a + \kappa_s)\rho\, dz, \tag{14}$$

with the condition that $\tau = 0$ at the top of the atmosphere, introducing the single scattering albedo, $\omega = \kappa_s/(\kappa_a + \kappa_s)$, and using the full fluxes, $F^{\pm} = G^{\pm} + \pi B$, we obtain

$$\frac{dF^+}{d\tau} = D(1 - \omega\alpha)F^+ - D\omega\beta F^- - D[1 - \omega(\alpha + \beta)]\pi B \tag{15}$$

and

$$\frac{dF^-}{d\tau} = D\omega\beta F^+ - D(1 - \omega\alpha)F^- + D[1 - \omega(\alpha + \beta)]\pi B, \tag{16}$$

which are the two-stream equations as given by Zdunkowski and Korb (1985). Using a diffusivity factor of 2 in these equations and in calculating the forward and backward scattering fractions from (12) and (13) we obtain Zunkowski's *Practical Improved Flux Method* (PIFM) (Zdunkowski *et al*, 1980); but for other values of the diffusivity factor we differ slightly from PIFM in that we have the diffusivity factor in (12) and (13), whereas Zdunkowski uses the forms appropriate to isotropic radiation, given by Wiscombe and Grams (1976). We therefore predict more asymmetric scattering than PIFM with diffusivity factors less than 2.

For the purposes of the subsequent discussion it is more convenient to retain the differential fluxes and to employ the quantity u, defined such that $u = 0$ at the top of the atmosphere and $du = -D\rho dz$, as a vertical coordinate. For our purposes, u is a more convenient variable than the optical depth since it is independent of the optical processes considered. In this formulation the two-stream equations are

$$\frac{dG^+}{du} = (\kappa_a + \kappa_s(1 - \alpha))G^+ - \kappa_s\beta G^- - \pi\frac{dB}{du} \tag{17}$$

and

$$\frac{dG^-}{du} = \kappa_s \beta G^+ - (\kappa_a + \kappa_s(1-\alpha))G^- - \pi \frac{dB}{du}. \qquad (18)$$

a. The Choice of the Diffusivity Factor

Since D (or $D_\pm$ in the more general form of the two-stream equations represented by (10)) has been defined from the differential radiance rather than the full radiance, there is no inherent requirement that it should be 2 in the optically thick limit. Indeed, in an atmosphere in which temperature decreases monotonically upwards we would expect $I' > 0$ on Ω_+, $I' < 0$ on Ω_-, with $|I'|$ increasing with $|\mu|$ and $I' = 0$ for all horizontal rays. Thus D would be expected to be less than 2. Indeed, the considerations determining the diffusivity factor are identical to those encountered in emissivity schemes (Goody and Yung, 1989). In an optically thick atmosphere, the differential radiance is proportional to the product of μ and the temperature gradient from which it follows that a diffusivity factor of 1.5 is appropriate. At very small optical depths in the atmosphere there is little radiation in any downward direction, so the differential radiance in these directions is close to $-B$ and $D_- \to 2$ as $\tau \downarrow 0$. On the other hand, as τ increases, the optical path in directions significantly inclined to the vertical quickly increases, causing the corresponding radiance rapidly to approach the Planckian value and D_- to fall below 2. As in the case of emissivity schemes, the value of 1.66, due originally to Elsasser (1943), generally gives good agreement with values calculated using more accurate representations of the angular dependence of the radiation. Whatever value is chosen, the optically thick limit is preserved.

However, though this value yields fairly accurate fluxes, it may not yield such accurate values of sensitivities to changes in the concentration of trace gases, and this may be significant in climate modelling. These points are illustrated in Table 1 which shows the net flux at the tropopause in a mid-latitude summer atmosphere in the spectral band from 990 cm^{-1} – 1120 cm^{-1} and its sensitivity to the presence of CFC-11 and CFC-12, calculated both with and without other gases, using three different schemes for treating the angular integration. When there are no gases present, the net flux at the tropopause is simply that coming up from the surface. When absorption by the CFCs alone is included, the atmosphere remains optically thin and a diffusivity factor of 2 accurately represents the departure of the radiance from the local Planckian value, so a two-stream approximation with this diffusivity factor gives a value for the change in the net flux due to the CFCs alone which agrees with that calculated using Gaussian quadrature; but a using diffusivity factor of 1.66 leads to an underestimate of this change. Considering absorption by ozone, water vapour and carbon dioxide but omitting the CFCs, the net flux is slightly more accurately calculated by assuming a diffusivity factor of 1.66; but though its sensitivity to the presence of CFCs is more accurately calculated with a diffusivity factor of 2, the actual value computed using Gaussian quadrature

lies between those obtained from the two-stream calculations. Across the whole infra-red region a diffusivity factor of 1.66 gives considerably more accurate values for the fluxes themselves, but leads to an underestimate of the sensitivity of about 10 %. These calculations make the point that the diffusivity factor depends on the actual radiance field, and hence on all the gases present and the atmospheric conditions: one should not conclude that a diffusivity factor of 2 is appropriate for calculating sensitivities to changes in the concentrations of CFCs simply on the basis that these gases themselves are optically thin.

3 The Treatment of Overlapping Absorption by Different Gases

The two-stream equations apply to monochromatic radiation. To calculate the flux in a wide spectral band with one absorbing gas, exponential sum fitting of transmissivities (ESFT) may be used: n quasi-monochromatic calculations of the flux within the band are performed using mass absorption coefficients $K_i, i = 1, \ldots, n$: the total flux is calculated by multiplying each quasi-monochromatic flux by a weight, w_i, and adding. The K_i and w_i are derived by fitting the transmission through the gas. To represent the random overlapping of gaseous absorption by two different gases, individually requiring respectively n_1 and n_2 ESFT terms, $n_1 n_2$ quasi-monochromatic calculations are required. The treatment of overlapping gaseous absorption is therefore computationally expensive. RG92 have proposed a more efficient approximate method of treating overlapping gaseous absorption which they call "Fast ESFT" (FESFT). They apply to the infra-red net fluxes a technique introduced in the shortwave region by Geleyn and Hollingsworth (1979), remarking that although they "could not find any consistent theoretical reason for the fact," it did give fairly accurate results. A theoretical analysis of FESFT in the infra-red region will now be presented, showing when its use is justified. A more efficient form of the algorithm will then be derived, and this will be used to construct a slightly different approximation for overlapping gaseous absorption which may be more suitable for application in a GCM.

a. A Derivation of FESFT

In FESFT, as described by RG92, one first calculates the *grey* net flux, $\bar{N}$, in a particular band, ignoring gaseous absorption but including any remaining optically grey processes operating within the band. Then for each gas, $j = 1, \ldots, m$, absorbing in the band the net flux, N_j, including the grey processes and absorption by that gas alone is calculated. An *effective transmission*, E_j, for the gas is then defined by $E_j = N_j/\bar{N}$, and the overall flux is approximated by multiplying

together all the effective transmissions and the grey flux:

$$N \approx \prod_{j=1}^{m} E_j . \bar{N}. \tag{19}$$

Only $1 + \sum_{i=1}^{m} n_i$ two-stream calculations are necessary, instead of $\prod_{j=1}^{m} n_i$ for random overlap.

In a given band, it is usually the case that one gas is the dominant absorber and that absorption by the other gases is somewhat weaker: we shall adopt the terms *major gas* and *minor gas* to distinguish between these absorbers. The right hand side of (19) may then be rewritten as a product the effective transmissions for the minor gases and the net flux for the major gas. This leads to a heuristic justification of FESFT. Consider an atmosphere initially containing only the optically grey components and the major gas. As the minor gases are successively included, the atmosphere becomes more opaque and the magnitude of the net flux is reduced. Normally, the E_j are positive and less than 1, so that as each minor gas is added, the net flux is relaxed towards its limiting value of zero. As we shall now see, this prescription applies naturally to the differential fluxes. Herein lies the contrast with the short-wave region of the spectrum: there, FESFT must be applied to the full fluxes, because it is they that tend to zero in the opaque limit, but in the infra-red it is the differential fluxes that tend to zero.

1) MATHEMATICAL DERIVATION FOR A PAIR OF ESFT TERMS

Consider a spectral band in which there is grey absorption and absorption by two gases, a major gas (*i. e.* the dominant absorber) and a second minor gas. Choosing a pair of ESFT exponents for each gas, K_1 for the major gas and K_2 for the minor, it will be shown that the downward differential flux for this pair of terms may be approximately factorized into contributions from each gas alone. Let $G_g^{\pm}$ denote the differential fluxes calculated considering only the grey processes: let $G_1^{\pm}$ be the differential fluxes with the chosen ESFT term for the major gas and the grey processes, and let $G_2^{\pm}$ be the differential fluxes calculated using the term chosen for the minor gas and the grey processes alone. Writing q_1 and q_2 for the mass mixing ratios of the gases, adjusted for any pressure or temperature scaling of the amount of absorber, and κ_a and κ_s to denote only the grey contributions to the mass extinction coefficients for absorption and scattering, the various downward differential fluxes obey the equations

$$\frac{dG_g^-}{du} = \kappa_s \beta G_g^+ - (\kappa_a + \kappa_s(1-\alpha))G_g^- - \pi \frac{dB}{du}, \tag{20}$$

$$\frac{dG_1^-}{du} = \kappa_s \beta G_1^+ - (\kappa_a + \kappa_s(1-\alpha) + K_1 q_1)G_1^- - \pi \frac{dB}{du} \tag{21}$$

and

$$\frac{dG_2^-}{du} = \kappa_s \beta G_2^+ - (\kappa_a + \kappa_s(1-\alpha) + K_2 q_2)G_2^- - \pi \frac{dB}{du}; \tag{22}$$

whilst the differential flux, G^-, including absorption by both gases, obeys the equation

$$\frac{dG^-}{du} = \kappa_s \beta G^+ - (\kappa_a + \kappa_s(1-\alpha) + K_1 q_1 + K_2 q_2) G^- - \pi \frac{dB}{du}. \tag{23}$$

Defining $\tilde{G}^\pm = (G_2^\pm / G_g^\pm) G_1^\pm$, and using (20) – (22),

$$\begin{aligned} \frac{d\tilde{G}^-}{du} &= \frac{G_2^-}{G_g^-}\frac{dG_1^-}{du} + \frac{G_1^-}{G_g^-}\frac{dG_2^-}{du} - \frac{G_2^- G_1^-}{(G_g^-)^2}\frac{dG_g^-}{du} \\ &= \kappa_s \beta \tilde{G}^+ - (\kappa_a + \kappa_s(1-\alpha) + K_1 q_1 + K_2 q_2)\tilde{G}^- - \pi \frac{dB}{du} \\ &+ \pi \frac{dB}{du}\left(1 - \frac{G_1^-}{G_g^-}\right)\left(1 - \frac{G_2^-}{G_g^-}\right) \\ &- \kappa_s \beta \tilde{G}^+ \left(1 - \frac{G_1^- G_g^+}{G_g^- G_1^+}\right)\left(1 - \frac{G_2^- G_g^+}{G_g^- G_2^+}\right). \end{aligned} \tag{24}$$

Thus, $\tilde{G}^-$ obeys an equation identical to that for the actual flux (Eq. 23), except for the error terms on the last two lines. In particular, the extinction is represented precisely and the errors are introduced into the representation of thermal emission and of scattering. The condition of no incident downward flux at the top of the atmosphere implies that $\tilde{G}^- = G^- = -\pi B_{00}$, where B_{00} is the Planck function for the temperature at the top of the atmosphere. Neglecting the second error term involving the scattering for the present, the error term in the temperature gradient causes $\tilde{G}^-$ to diverge from G^- on descending into the atmosphere, but while $G_2^- \approx G_g^-$ it remains small and may consistently be ignored, until a level is reached where it becomes comparable to the retained terms. More formally, comparing the error term with the gradient of the Planck function, we require that $|(1-G_1^-/G_g^-)(1-G_2^-/G_g^-)| \ll 1$. If the major gas absorbs strongly, $|1-G_1^-/G_g^-|$ will not be small and the condition implies that $|1-G_2^-/G_g^-|$ must be small, which in turn implies that the minor gas must be a weak absorber. Having established that the minor gas must absorb weakly, it remains to be checked that the error term is less than the smallest retained term, that representing extinction by the minor gas. This condition is harder to analyse, but it implies that consistency will be lost when u exceeds $|B/(dB/du)|$, the scale for variations of the Planck function. In practice, these conditions may be found excessively stringent: they show when the approximation ceases to predict changes in G^- consistently, but in atmospheric calculations the greatest changes in G^- will often have occurred above this level and the actual values of G^- may still be acceptably accurate. A non-scattering isothermal atmosphere constitutes an important special case wherein the two-stream equation for the downward differential flux reduces to one of transmission and the error term vanishes.

The contributions form each pair of ESFT terms may be summed to give the

total differential flux in the band:

$$G^- = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} w_i^{(1)} w_j^{(2)} \frac{G_j^{(2)-}}{G_g^-} G_i^{(1)-} = \frac{G^{(2)-}}{G_g^-} G^{(1)-} = \frac{G^{(1)-}}{G_g^-} \cdot \frac{G^{(2)-}}{G_g^-} . G_g^- , \qquad (25)$$

where the bracketed superscripts refer to the major and minor gases.

The upward flux may be approximated in a similar way, but in this case, the approximation is most accurate at the bottom of the atmosphere and, as will be seen in the example below, the large temperature gradient in the stratosphere leads to significant errors in the heating rates there. Provided that $G_2^{\pm} \approx G_g^{\pm}$, the error term involving the scattering will also be small and the approximation may be applied in a scattering atmosphere.

It is not so easy to derive a consistent analogue of (25) for the net fluxes, but since in practice G^- is often the dominant contribution to N, (25) may be applied to the net fluxes, thus recovering the formulation of RG92.

2) AN EXAMPLE

Figure 1 shows the net flux and the cooling rate obtained by applying FESFT in the spectral region 1200 – 1500 cm^{-1}, where absorption by water vapour overlaps with weaker absorption by nitrous oxide and methane. (The dotted line labelled "Equivalent Extinction" will be discussed later). The upper figure shows the net flux calculated including absorption by water vapour alone, that calculated including randomly overlapping absorption by all three gases, and that calculated representing the overlapping absorption using FESFT. Whether used in the original form proposed by RG92 for the net fluxes, or in the modified form for the differential fluxes given above, FESFT gives the effect of the weaker absorption on the net flux with reasonable accuracy, especially at low levels in the atmosphere. Applying FESFT to the differential fluxes yields marginally better results than applying it to the net flux alone, except at very high levels in the atmosphere, where this form of the approximation predicts a spurious divergence of the net flux. The reason for this failure is readily understood from consideration of the error terms in the counterpart of (24) for the upward flux. At such high levels, the gradient of the Planck function is very large: in the case of the downward flux this does not introduce a significant error, since factors multiplying $\pi\, dB/du$ in the first error term of (24) are very small; but in the case of the upward flux the corresponding factors are larger, with the error term introducing a significant divergence into $\tilde{G}^+$. The lower figure shows that cooling rates are rather less well predicted by both forms of FESFT, with the spurious divergence of the upward flux near the top of the atmosphere obtained with the differential form of FESFT leading to serious errors at these levels.

b. Clear-Sky FESFT

Though applying FESFT to the next fluxes gave fairly good results in the above example, the method is not completely robust. For example, when a cloud lies above a colder surface the grey net flux may be close to zero and of either sign. The effective transmissions may then be large in magnitude and lead to unphysical results when multiplied together. By deriving the factor representing the effect of the minor gas by which the differential flux is relaxed towards zero differently, a more robust approximation may be derived. This factor may be found from a calculation involving just the minor gas. Suppose now that $G_v^{\pm}$ are the fluxes *in vacuo*,

$$\frac{dG_v^-}{du} = -\pi \frac{dB}{du}, \tag{26}$$

that $G_1^{\pm}$ are as before, but that $G_2^{\pm}$ now represent the solution with the minor gas alone:

$$\frac{dG_2^-}{du} = -K_2 q_2 G_2^- - \pi \frac{dB}{du}. \tag{27}$$

G_2^- is therefore determined from a calculation in which the effects of other atmospheric constituents, such as clouds, are ignored and this will be termed a *clear-sky* calculation. Now defining $\tilde{G}^{\pm} = (G_2^{\pm}/G_v^{\pm})G_1^{\pm}$, we find that

$$\begin{aligned} \frac{d\tilde{G}^-}{du} &= \kappa_s \beta \tilde{G}^+ - (\kappa_a + \kappa_s(1-\alpha) + K_1 q_1 + K_2 q_2)\tilde{G}^- - \pi \frac{dB}{du} \\ &+ \pi \frac{dB}{du}\left(1 - \frac{G_1^-}{G_v^-}\right)\left(1 - \frac{G_2^-}{G_v^-}\right) - \kappa_s \beta \tilde{G}^+ \left(1 - \frac{G_2^- G_v^+}{G_v^- G_2^+}\right). \end{aligned} \tag{28}$$

Again, if absorption by the second gas is weak, the error terms may consistently be neglected and $\tilde{G}^-$ will be a good approximation to G^-. This new form of the approximation will be referred to as as *Clear-sky FESFT* (CFESFT) to distinguish it from the previous form of RG92, to which it reduces in the case of gaseous absorption alone. Generally, however, CFESFT is computationally more efficient, since only clear-sky calculations for the minor gas are required. It is more robust than FESFT for the calculation of G^- (and hence for the calculation of N), since, if the minor gas is a weak absorber, it alone will not be able to generate sufficient downward flux to make G^- pass through zero. Unfortunately, it may still fail in the calculation of G^+ when there is an inversion near the surface, for then there will be a level where $G_v^+ = 0$ though generally G_2^+ will be non-zero, giving an infinite effective transmission.

Despite these advantages, CFESFT gives rather poorer results for the net flux above clouds in atmospheric profiles without an inversion at the surface. Figure 2 shows the net flux for the same example as in Figure 1, but including an ice cloud with an optical depth, based on geometrical optics, of 1.91. Below the cloud both FESFT and CFESFT are about equally accurate, but above the cloud CFESFT over-estimates the reduction in the net flux. This behaviour occurs

because the cloud partially decouples the radiation fields above and below it; whereas the effective transmissions for methane and nitrous oxide above the cloud are calculated ignoring the effects of the cloud and therefore depend too strongly on conditions below it.

The derivations of FESFT and CFESFT given above hinged upon the assumption that only the major gas in the band absorbed strongly. In the spectral region 550 – 800 cm^{-1} both carbon dioxide and water vapour are strong absorbers, and it is not clear that the approximations are still justified. Nevertheless, as implied by RG92 and as shown in Figure 3, these approximations yield fairly accurate heating rates. Table 2, however, shows that using FESFT for the net fluxes reduces the OLR in this region by almost 3 Wm^{-2}, and that the net flux at the tropopause is 1.6 Wm^{-2} too low, although the sensitivities of these fluxes to a doubling of the concentration of carbon dioxide are fairly close to the values obtained from explicit representation of the overlap. The low value of the OLR arises from an excessive reduction in the net flux high in the atmosphere by the effective transmission for water vapour (which must be taken as the minor gas in the region 550-800 cm^{-1}). Absorption by carbon dioxide largely decouples the radiation field low in the atmosphere from that at higher levels, an effect which is not accounted for when calculating the effective transmission for water vapour.

c. *Equivalent Extinction*

The excessive reduction in the OLR discussed above seems to be an inherent feature of any method based upon the relaxation of the net flux towards zero. We will now discuss an alternative method of treating the gaseous overlaps, which makes use of CFESFT, but mitigates this problem. This will be referred to as the method of *Equivalent Extinction* (EE).

Consider again the equation for the downward differential flux (18) and now let κ_a and κ_s denote the mass extinction coefficients of absorption and scattering due to any arbitrary collection of optical processes, excluding absorption by one particular minor gas. Let the solution ignoring the minor gas be $G_0^{\pm}$ so that

$$\frac{dG_0^-}{du} = \kappa_s \beta G_0^+ - (\kappa_a + \kappa_s(1-\alpha))G_0^- - \pi \frac{dB}{du}. \tag{29}$$

Let the ESFT exponents of the minor gas be $K_i, i = 1, \ldots n$, with corresponding weights $w_i, i = 1, \ldots n$, and let $G_i^{\pm}$ be the differential fluxes calculated by including absorption from the ith term for the minor gas as well. The overall differential flux, G^-, is then given by $G^- = \sum_{i=1}^{n} w_i G_i^-$, so that

$$\begin{aligned} \frac{dG^-}{du} &= \sum_{i=1}^{n} w_i \frac{dG_i^-}{du} \qquad (30) \\ &= \sum_{i=1}^{n} w_i \left[\kappa_s \beta G_i^+ - (\kappa_a + \kappa_s(1-\alpha) + K_i q) G_i^- - \pi \frac{dB}{du} \right] \end{aligned}$$

$$= \kappa_s \beta G^+ - (\kappa_a + \kappa_s(1-\alpha) + \bar{K}^- q) G^- - \pi \frac{dB}{du}$$

where q is the mass mixing ratio of the minor gas and we define

$$\bar{K}^- = \sum_{i=1}^{n} w_i K_i G_i^- \Big/ \sum_{i=1}^{n} w_i G_i^- . \tag{31}$$

$\bar{K}^-$ is an equivalent (grey) extinction representing the effect of the minor gas on downward radiation. In order to determine $\bar{K}^-$ precisely it would be necessary to calculate each G_i^-, and this would require the solution of the full problem. However, by approximating G_i^- using CFESFT, the equation for $\bar{K}^-$ may be simplified very effectively; for then, $G_i^- \approx (G_{v,i}^-/G_v^-)G_0^-$, where G_v^- represents the solution *in vacuo* and $G_{v,i}^-$ represents the flux calculated including only the ith ESFT term for the minor gas. Hence,

$$\bar{K}^- = \sum_{i=1}^{n} w_i K_i G_{v,i}^- \Big/ \sum_{i=1}^{n} w_i G_{v,i}^- ; \tag{32}$$

so that $\bar{K}^-$ can be computed directly from a calculation involving the minor gas alone and, to the extent that CFESFT is valid, is independent of the ESFT term of the major gas. The total flux in the band is then obtained by performing a full calculation for each ESFT term of the major gas, but representing the effects of the minor gas as a grey process through the corresponding equivalent extinction.

A similar argument may be employed for the upward differential flux, yielding an equivalent extinction, $\bar{K}^+$. Generally, $\bar{K}^+ \neq \bar{K}^-$, but this is consistent with the more general formulation of the two-stream equations where $D_+ \neq D_-$. However, in a GCM, where only N may be available, and in the standard form of the two-stream equations, where $D_+ \equiv D_- = D$, it would be convenient to use the single equivalent extinction based on the net fluxes:

$$\bar{K} = \sum_{i=1}^{n} w_i K_i N_{v,i} \Big/ \sum_{i=1}^{n} w_i N_{v,i} . \tag{33}$$

Tests show that an equivalent extinction defined in this way will perform satisfactorily, probably because G^- usually represents the principal contribution to N. Considering the overlap of methane and nitrous oxide with water vapour in the spectral region 1200 – 1500 cm^{-1}, Figure 1 shows that EE predicts the net flux more accurately than FESFT in the upper atmosphere, although not quite so well in the lower atmosphere, and a similar signature is seen in Figure 3 for the overlap between carbon dioxide and water vapour between 550 and 800 cm^{-1}, where carbon dioxide is taken as the major gas. Table 2 shows that EE gives much more accurate results for the upward flux at the top of the atmosphere in this spectral region than FESFT. This concentration of errors in the lower atmosphere appears to be a characteristic error of EE and arises because the weighting with the

net flux in the definition of $\bar{K}$ results in less importance being given to stronger exponents, so that $\bar{K}$ will usually be too small. Indeed, in an optically thick regime, considering only gaseous absorption, $G^- \sim -(\pi/(K_1 q_1 + K_2 q_2))(dB/du)$, so if absorption by the major gas dominates for all ESFT terms of the minor gas $\bar{K} \approx K_g \equiv \sum_{i=1}^{n} w_i K_i$, the grey extinction of the minor gas. This raises the possibility of modifying the definition of $\bar{K}$ in such a way that it approaches K_g in the opaque limit. Some experiments in this direction have been carried out, but the modifications tried so far have all degraded the performance of the approximation in the optically thin limit. It may also be noted that it is not essential to use CFESFT itself to define the equivalent extinction: all that is required is that the equivalent extinction may be evaluated more quickly than by calculating the fluxes using full random overlap and some other approximation than CFESFT might yield better results.

4 Discussion

Though emissivity schemes are most commonly used for the calculation of infra-red fluxes in GCMs, the quadratic dependence of their computational cost on the number of atmospheric layers makes them potentially less suitable than two-stream approximations, the computational costs of which depend linearly on the number of layers, when high vertical resolutions are employed. Two-stream approximations also permit the effects of scattering to be included.

Here, it has been emphasised that at infra-red wavelengths the two-stream equations may be analysed effectively by considering differential radiances and fluxes (differences between the total radiance or flux and the Planckian radiance or flux); for at these wavelengths the competition between absorption and emission acts to reduce the differential fluxes, and in an optically dense medium to establish a regime in which the differential fluxes are almost zero. The two-stream approximation itself may be viewed naturally as an approximation for the differential fluxes. In this formulation the diffusivity factor and the forward and backward scattering fractions are related to the differential radiance, not the total radiance. The diffusivity factor is not required to be 2 to maintain physical consistency in the optically thick limit. Indeed, the criteria for choosing an appropriate value for this parameter are identical to those applying in emissivity schemes, and the standard value of 1.66 was seen to give a good representation. However, with applications in climate studies in mind, it was also noted that a diffusivity factor which yields accurate values of the fluxes themselves may not yield such accurate values of the sensitivities of these fluxes to changes in the concentrations of trace gases.

Representing gaseous transmissions using ESFT, fluxes may be calculated by performing a number of quasi-monochromatic calculations and summing the re-

sults, but to represent randomly overlapping gaseous absorption explicitly using this method involves considerable computational expense. By considering the differential fluxes, approximate methods for representing overlapping absorption have been derived, including the method of Fast ESFT (FESFT) proposed by RG92 and the new methods of Clear-sky FESFT (CFESFT) and Equivalent Extinction (EE). Very substantial savings in computing time are made possible by these methods and they therefore find their principal application in GCMs: in this context, formulations in the net flux alone are to be preferred for their computational efficiency. The accuracy of these methods was assessed by performing simple illustrative calculations.

CFESFT is to be preferred to the original form of FESFT presented by RG92 where speed and robustness are important, though it is less accurate above cloudy layers. EE is faster and more robust than FESFT and gives more accurate fluxes at higher levels in the atmosphere. EE generally introduces comparable errors into the cooling rate, though with a different vertical distribution from those produced by FESFT and CFESFT, which approximate the cooling rate more accurately in the lower atmosphere, whereas EE is more accurate at higher levels. There may be a logical advantage in the use of EE since it involves the direct approximation of the absorption by the minor gas, rather than an adjustment of the total fluxes, and thus avoids an implicit adjustment of other radiative processes: this may be desirable in some applications, such as studies of the climatic effects of clouds. On balance these considerations have led us to prefer EE for climate integrations in the Hadley Centre GCM, using the radiation code described by Edwards and Slingo (1995).

Whilst ESFT has been used to represent gaseous transmissions in the present study, it would have been possible to use correlated k-distributions: in this method fluxes are also calculated by summing the results of a number of quasi-monochromatic calculations and the explicit representation of randomly overlapping absorption is computationally expensive. For use in a GCM an approximate representation of the overlap is required. Lacis and Oinas (1991) describe a reblocking of the joint k-distribution and Fu and Liou (1992) describe a method of representing the overlap by introducing a k-distribution for the overlapping absorption which is parametrized in terms of the ratio of the mixing ratios of the two gases. Though no attempts have been made to apply FESFT or EE in the context of the method of correlated-k, there would appear to be no reason in principle why they should not be applicable, but it is not possible to say in advance how well they would compare against the methods of Lacis and Oinas and Fu and Liou in terms of speed and accuracy.

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Captions for Figures

Figure 1 The net upward flux and the cooling rate in the spectral region 1200 – 1500 cm^{-1} in a mid-latitude summer atmosphere including absorption by water vapour only and including absorption by nitrous oxide at 0.31 ppmv and by methane at 1.72 ppmv. The absorption is treated assuming explicit random overlap, by the method of Fast ESFT (FESFT) for the differential and the net fluxes and by Equivalent Extinction (EE). The actual fluxes and heating rates are shown in the left-hand panels and the errors of the approximate methods relative to the calculation with explicit random overlap are shown in the right-hand panels.

Figure 2 As Figure 2, but including a cloud composed of a poly-dispersion of ice spheres with an effective radius of 40 μm and a mass mixing ratio of 5×10^{-5} lying between 200 and 300 hPa. The absorption is treated assuming explicit random overlap and using the methods of Fast ESFT (FESFT) and Clear-sky Fast ESFT (CFESFT) for the net fluxes and using Equivalent Extinction (EE). Only net fluxes are shown.

Figure 3 The net upward flux and the cooling rate in the spectral region 550 – 800 cm^{-1} in a mid-latitude summer atmosphere calculated including absorption by carbon dioxide at 353 ppmv alone and including absorption by water vapour, ozone and nitrous oxide at 0.31 ppmv as well. The absorption is treated assuming explicit random overlap, by the method of Fast ESFT (FESFT) for the differential and the net fluxes and by Equivalent Extinction (EE). The actual fluxes and heating rates are shown in the left-hand panels and the errors of the approximate methods relative to the calculation with explicit random overlap are shown in the right-hand panels.

Tables

Table 1: The net flux at the tropopause, N_T, in a mid-latitude summer atmosphere in the spectral region from 990 – 1120 cm^{-1} and the sensitivity of this flux, ΔN_T, to the presence of CFC-11 at 280 pptv and CFC-12 at 484 pptv. In the first set of results there are no background gases, so that CFC-11 and CFC-12 are the only absorbers considered. In the second set, the background absorption by water vapour, carbon dioxide and ozone is included. Values are calculated using 10-point Gaussian quadrature and two-stream approximations with diffusivity factors, D, of 2 and 1.66.

	No Background Gases			All Gases Present		
	Gaussian	$D = 2.0$	$D = 1.66$	Gaussian	$D = 2.0$	$D = 1.66$
N_T	32.837	32.837	32.837	30.183	29.933	30.351
ΔN_T	0.110	0.110	0.092	0.090	0.096	0.081

Table 2: The net upward flux at the tropopause, N_T, the out-going longwave radiation, OLR, and the net upward flux at the surface, N_S, in a mid-latitude summer atmosphere across the spectral band 550 – 800 cm^{-1}. Absorption by carbon dioxide at 353 ppmv, water vapour, ozone and nitrous oxide at 0.310 ppmv is included. The sensitivities of these fluxes to a doubling of the concentration of carbon dioxide are also shown. Fluxes are calculated assuming explicit random overlap, using Fast ESFT (FESFT) and using Equivalent Extinction (EE).

	Random Overlap		FESFT		EE	
	353 ppmv	Sens.	353 ppmv	Sens.	353 ppmv	Sens.
N_T	50.66	-5.18	49.09	-5.15	50.63	-5.17
OLR	63.38	-2.33	60.67	-2.62	63.77	-2.33
N_S	4.67	-0.62	4.48	-0.61	4.92	-0.65

Figures

Figure 1.

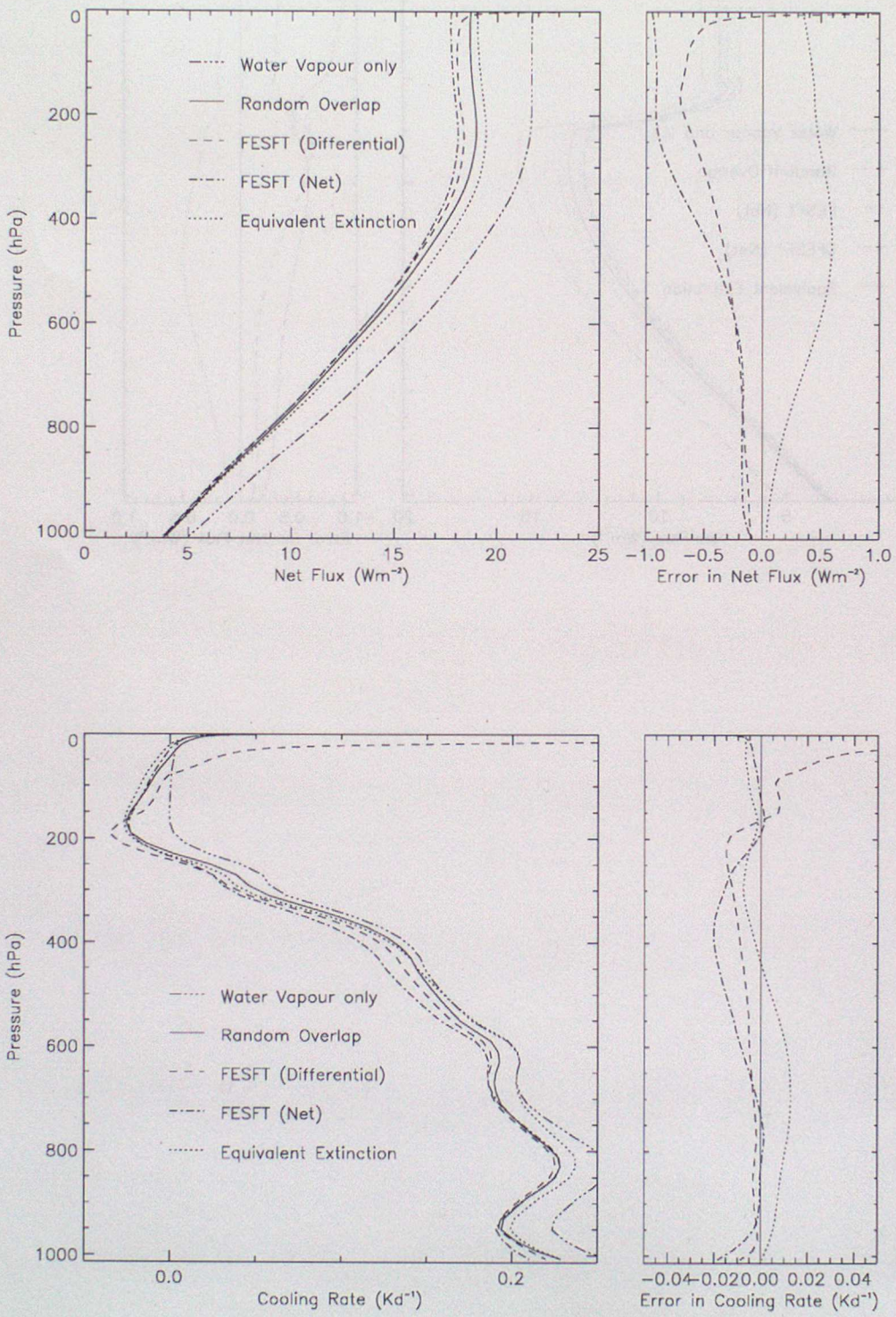


Figure 2.

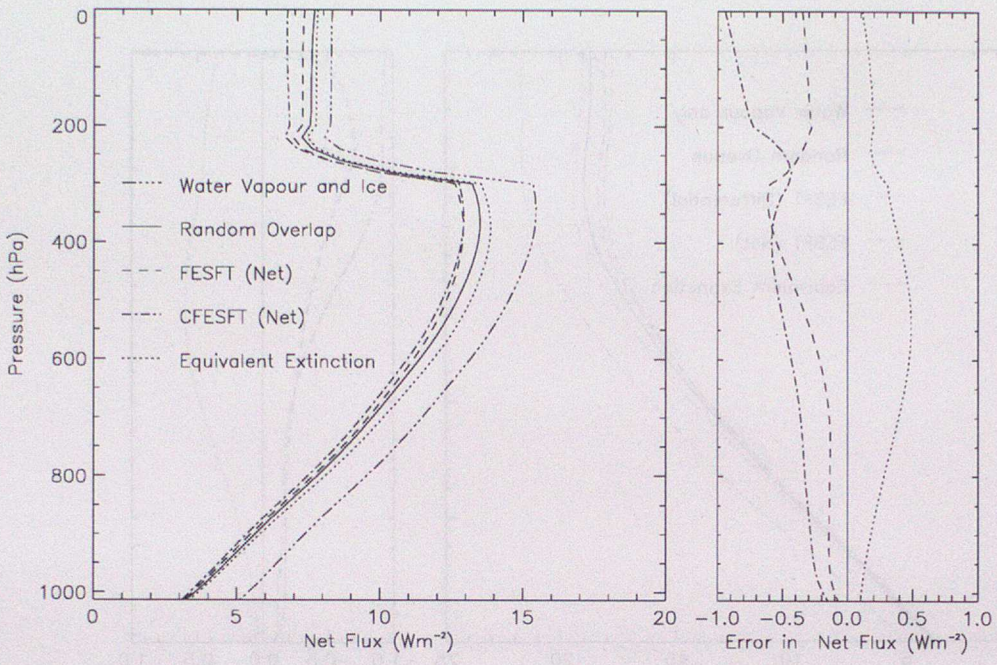
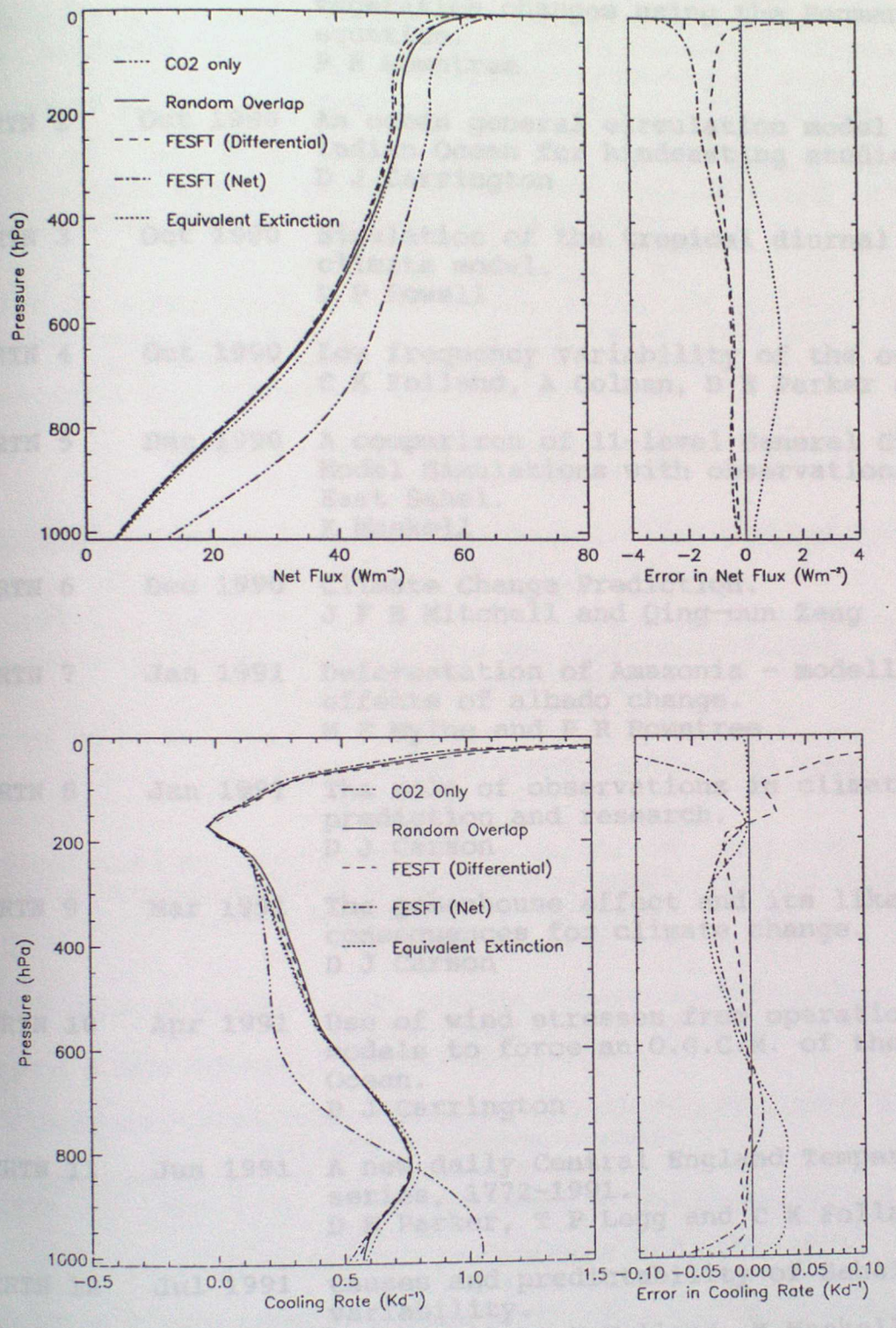

Pressure (hPa)
0
200
400
600
800
1000
Water Vapour and Ice
Random Overlap
FESFT (Net)
CFESFT (Net)
Equivalent Extinction
0
5
10
15
20
Net Flux (Wm⁻²)
−1.0
−0.5
0.0
0.5
1.0
Error in Net Flux (Wm⁻²)

Figure 3.


CO2 only
Random Overlap
FESFT (Differential)
FESFT (Net)
Equivalent Extinction
Pressure (hPa)
Net Flux (Wm⁻²)
Error in Net Flux (Wm⁻²)
CO2 Only
Random Overlap
FESFT (Differential)
FESFT (Net)
Equivalent Extinction
Pressure (hPa)
Cooling Rate (Kd⁻¹)
Error in Cooling Rate (Kd⁻¹)